Convergence rate of Padé-type approximants for Stieltjes functions

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Abstract

For a wide class of Stieltjes functions we estimate the rate of convergence of Padé-type approximants when the number of fixed poles represents a fixed proportion with respect to the order of the rational approximant.

Keywords: Orthogonal polynomials; Padé-type approximation

1. Introduction

Let $\gamma > 1$, by f_γ we denote a continuous almost everywhere positive function on the real line such that

$$\lim_{|x|\to\infty} f_\gamma(x)|x|^{-\gamma} = 1. \tag{1}$$

In [9], Rakhmanov studied the asymptotic behavior of the sequence $h_n(d\rho_\gamma;.)$ of orthonormal polynomials with respect to

$$d\rho_\gamma(x) = \exp\{-f_\gamma(x)\}\,dx, \quad x \in \mathbb{R}. \tag{2}$$

(Within this class of measures, of particular interest are the so-called Freud weights

$$dw_\gamma(x) = \exp\{-|x|^\gamma\}dx$$

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and their orthogonal polynomials.) He proved that

$$\lim_{n\to\infty} \frac{\log |h_n(\mathrm{d}\rho_\gamma; z)|}{n^{1-\gamma^{-1}}} = D(\gamma)|\mathrm{Im}\, z|, \tag{3}$$

where this limit is uniform on compact subsets of $\mathbb{C}\setminus\mathbb{R}$,

$$D(\gamma) = \frac{\gamma}{\gamma - 1}\left[\frac{\Gamma((\gamma+1)/2)}{\Gamma(\gamma/2)}\right]^{(1/\gamma)},$$

and $\Gamma(.)$ denotes the Gamma function. Set

$$\hat{\rho}_\gamma(z) = \int \frac{\mathrm{d}\rho_\gamma(x)}{z - x}.$$

Let π_n denote the nth diagonal Padé approximant with respect to $\hat{\rho}_\gamma$. From Rakhmanov's result it is not hard to deduce that

$$\lim_{n\to\infty} \frac{\log |\hat{\rho}_\gamma(z) - \pi_n(z)|}{n^{1-\gamma^{-1}}} \leqslant -2D(\gamma)|\mathrm{Im}\, z|, \tag{4}$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. We aim to obtain similar results when instead of Padé approximants, Padé-type approximants are used.

Let l_n^2 be a polynomial of degree $m(n)$ and $0 \leqslant m(n) \leqslant n$. We define the nth Padé-type approximants of $\hat{\rho}_\gamma$ with fixed poles at the zeros of l_n^2 as the unique rational function

$$r_n = \frac{p_n}{q_n l_n^2},$$

where p_n and q_n are polynomials which satisfy

- $\deg p_n \leqslant n - 1$, $\deg q_n \leqslant n - m(n)$, $q_n \not\equiv 0$,
- $(q_n l_n^2 \hat{\rho}_\gamma - p_n)(z) = \mathrm{O}(1/(z^{n-m(n)+1}))$, as $z = \mathrm{i}x \to \infty$, $x > 0$.

It is easy to prove (see, e.g., [4]) that

$$0 = \int x^\nu q_n(x) l_n^2(x) \mathrm{d}\rho_\gamma(x), \quad \nu = 0, \ldots, n - m(n) - 1, \tag{5}$$

$$(\hat{\rho}_\gamma - r_n)(z) = \frac{1}{(q_n l_n)^2(z)} \int \frac{(q_n l_n)^2(x)\mathrm{d}\rho_\gamma(x)}{z - x}. \tag{6}$$

If $m(n) = 0$, then r_n is the nth diagonal Padé approximant with respect to $\hat{\rho}_\gamma$. If $m(n) = n$, all the poles of the rational approximant are fixed.

In recent years (see, e.g., [1–7]), the rate of convergence of Padé-type and multipoint Padé-type approximants has been studied when the measure defining the function has compact support. We will show that results of type (4) take place for Padé-type approximants when the support of the measure is unbounded. To this end, we will restrict the type of polynomials which carry as their zeros the fixed poles of the Padé-type approximants. In the sequel, l_n denotes the orthonormal polynomial of degree $m(n)/2$ with respect to the Freud measure $\mathrm{d}w_\beta(x)$ introduced above. Unless otherwise stated, we take $\gamma > \beta > 1$.

We prove

Theorem 1. *Let l_n denote the orthonormal polynomial of degree $m(n)/2$ with respect to the Freud measure $\mathrm{d}w_\beta(x)$ where $1<\beta<\gamma$. Let r_n denote the nth Padé-type approximant of $\hat{\rho}_\gamma$ with fixed poles at the zeros of l_n^2 and assume that*

$$\lim_{n\to\infty}\frac{m(n)}{n}=\theta\in[0,1).$$

Then

$$\limsup_{n\to\infty}\frac{\log|\hat{\rho}_\gamma(z)-r_n(z)|}{n^{1-\gamma^{-1}}}\leqslant -2(1-\theta)^{1-\gamma^{-1}}D(\gamma)|\operatorname{Im}z|,\tag{7}$$

where convergence takes place uniformly on each compact subset of $\mathbb{C}\backslash\mathbb{R}$.

The paper is divided as follows. In Section 2, we give some auxiliary results. Section 3 is devoted to the proof of the theorem stated above and some comments.

2. Auxiliary results

Let $\mathrm{d}\rho$ be a finite positive Borel measure on $\mathbb{R}$, with an infinite number of points in its support and finite moments. Denote

$$K_j(\mathrm{d}\rho,z)=\sup_{p\in\Pi_j,\,p\neq 0}\frac{|p^2(z)|}{\int|p^2(x)|\mathrm{d}\rho(x)},\tag{8}$$

where Π_j is the set of all polynomials of degree $\leqslant j$.

If $\mathrm{d}\rho=l_n^2\,\mathrm{d}\rho_\gamma$ we denote

$$K_{n,j}(z)=K_j(\mathrm{d}\rho,z).$$

Lemma 2.1. *There exist constants $D>0$ and $\alpha\in\mathbb{R}$ such that*

$$\mathrm{D}n^\alpha K_j(\mathrm{d}\tilde{\rho}_\gamma,z)\leqslant K_{n,j}(z)\leqslant K_j(l_n^2\mathrm{d}\rho_\gamma|_{(-n^{1/\gamma},n^{1/\gamma})},z),\tag{9}$$

where $\mathrm{d}\tilde{\rho}_\gamma(x)=\exp\{-(f_\gamma(x)-|x|^\beta)\}\,\mathrm{d}x$, $1<\beta<\gamma$, and $l_n^2\,\mathrm{d}\rho_\gamma|_{(-n^{1/\gamma},n^{1/\gamma})}$ is the restriction of the measure $l_n^2\,\mathrm{d}\rho_\gamma$ to $(-n^{1/\gamma},n^{1/\gamma})$.

Proof. From (8) the inequality on the right side of (9) follows directly. On the other hand from Corollary 1.4 in [7] there exist constants $D_1>0$ and $\alpha_1\in\mathbb{R}$ such that

$$l_n^2(x)\exp\{-|x|^\beta\}\leqslant D_1(m(n)+1)^{\alpha_1},\quad x\in\mathbb{R}.$$

Since $0\leqslant m(n)\leqslant n$, we obtain

$$l_n^2(x)\exp(-|x|^\beta)\leqslant D_2n^{\alpha_1}.$$

Thus, if $p \in \Pi_j$ and $p \not\equiv 0$, we get

$$\frac{|p^2(z)|}{\int |p^2(x)| l_n^2(x)\,\mathrm{d}\rho_\gamma(x)} \geqslant \frac{|p^2(z)|}{D_2 n^{\alpha_1} \int |p^2(x)|\,\mathrm{d}\tilde{\rho}_\gamma(x)}$$

and the proof is concluded. □

Lemma 2.2. *Let K be a compact subset of $\mathbb{C}\backslash\mathbb{R}$, $1<\beta<\gamma$, and $\lim(m(n)/n)=\theta$. Then*

$$\liminf_{n\to\infty} \frac{\log|q_n|(z)}{n^{1-1/\gamma}} \geqslant (1-\theta)^{1-1/\gamma} D(\gamma)|\mathrm{Im}\, z|$$

uniformly on K, where q_n is the $(n-m(n))$th orthonormal polynomial with respect to $l_n^2\,\mathrm{d}\rho_\gamma$ and l_n denotes the orthonormal polynomial of degree $m(n)/2$ with respect to the Freud measure $\mathrm{d}w_\beta(x)$.

Proof. Let t_k be the kth orthonormal polynomial with respect to $\mathrm{d}\rho$. From the general theory of orthogonal polynomials, we know (see [5]) that

$$K_j(\mathrm{d}\rho, z) = \sum_{k=0}^{j} |t_k(z)|^2 \geqslant |t_j(z)|^2, \quad z \in \mathbb{C}, \tag{10}$$

$$K_{j-1}(\mathrm{d}\rho, z) = \frac{\tau_{j-1}}{\tau_j} \frac{t_j(z)t_{j-1}(\bar{z}) - t_j(\bar{z})t_{j-1}(z)}{z-\bar{z}},$$

where $z \in \mathbb{C}\backslash\mathbb{R}$ and τ_j is the leading coefficient of t_j. Thus, with the aid of (10), we obtain

$$\begin{aligned} K_{j-1}(\mathrm{d}\rho, z) &= \frac{\tau_{j-1}}{\tau_j} \frac{\mathrm{Im}(t_j(z)\overline{t_{j-1}(z)})}{\mathrm{Im}\, z} \\ &\leqslant \frac{\tau_{j-1}}{\tau_j} \frac{|t_j t_{j-1}(z)|}{|\mathrm{Im}\, z|} \\ &\leqslant \frac{\tau_{j-1}}{\tau_j} \frac{|t_j(z)| K_{j-1}^{1/2}(z)}{|\mathrm{Im}\, z|}. \end{aligned}$$

This inequality yields

$$K_{j-1}(\mathrm{d}\rho, z) \leqslant \frac{\tau_{j-1}^2}{\tau_j^2} \frac{|t_j(z)|^2}{|\mathrm{Im}\, z|^2}, \tag{11}$$

therefore,

$$K_j(\mathrm{d}\rho, z) = K_{j-1} + |t_j(z)|^2 \leqslant \left[\frac{\tau_{j-1}^2}{\tau_j^2 |\mathrm{Im}(z)|^2} + 1\right] |t_j(z)|^2. \tag{12}$$

On the other hand,

$$\frac{1}{\tau_j^2} = \inf_{P=z^j+\cdots} \int |P^2(x)|\,\mathrm{d}\rho(x)$$
$$\leqslant \int \left| x\frac{t_{j-1}(x)}{\tau_{j-1}} \right|^2 \mathrm{d}\rho(x),$$

or what is the same

$$\frac{\tau_{j-1}^2}{\tau_j^2} \leqslant \int |xt_{j-1}|^2\,\mathrm{d}\rho(x).$$

If $\mathrm{d}\rho(x)$ satisfies (2), there exist constants D_1, D_2, $D_3 > 0$ such that for all $k \in \mathbb{N}$ and $p \in \Pi_k$, we have (see Theorem 2.6 in [8])

$$\int |p^2(x)|\,\mathrm{d}\rho(x) \leqslant D_2 \int_{-D_1 k^{1/\gamma}}^{D_1 k^{1/\gamma}} |p(x)|^2\,\mathrm{d}\rho(x),$$

in particular,

$$\frac{\tau_{j-1}^2}{\tau_j^2} \leqslant D_2 \int_{-D_1 j^{1/\gamma}}^{D_1 j^{1/\gamma}} |xt_{j-1}(x)|^2\,\mathrm{d}\rho(x),$$
$$\leqslant D_3 j^{2/\gamma}. \tag{13}$$

Take $\mathrm{d}\rho(x) = \mathrm{d}\tilde{\rho}_\gamma(x) = \exp\{-(f_\gamma(x) - |x|^\beta)\}\,\mathrm{d}x$. Since $1 < \beta < \gamma$, the function $f_\gamma(x) - |x|^\beta$ satisfies (1). Using (3), (10), (12), and (13) one obtains

$$\lim_{n\to\infty} \frac{\log K_n(\mathrm{d}\tilde{\rho}_\gamma, z)}{n^{1-1/\gamma}} = 2D(\gamma)|\mathrm{Im}(z)|. \tag{14}$$

This result appears in [9], Lemma 4.

For $\mathrm{d}\rho(x) = l_n^2(x)\,\mathrm{d}\rho_\gamma(x)$ and $j = n - m(n)$, (12) gives

$$K_{n,n-m(n)}(z) \leqslant \left[\frac{\tau_{n,n-m(n)-1}^2}{\tau_{n,n-m(n)}^2} + 1 \right] |q_n(z)|^2, \tag{15}$$

where q_n is the $(n-m(n))$th orthonormal polynomial with respect to $|l_n|^2\,\mathrm{d}\rho_\gamma$ and $\tau_{n,n-m(n)}$ its leading coefficient. Notice that, infinite–finite range L_2 estimates give as above

$$\frac{\tau_{n-m(n)-1}^2}{\tau_{n-m(n)}^2} \leqslant D_4 n^{2/\gamma}. \tag{16}$$

From the first inequality in (9), (14)–(16), we obtain

$$\liminf_{n\to\infty} \frac{\log|q_n(z)|}{n^{1-1/\gamma}} \geqslant \lim_n \left(\frac{n-m(n)}{n}\right)^{1-1/\gamma} \frac{\log|K_{n-m(n)}(\mathrm{d}\tilde{\rho}_\gamma, z)|}{2(n-m(n))^{1-1/\gamma}}$$
$$= (1-\theta)^{1-1/\gamma} D(\gamma)|\mathrm{Im}(z)|$$

and the proof is finished. □

3. Proof of Theorem 1

Let K be a compact subset of $\mathbb{C}\backslash\mathbb{R}$, then there exists $D_1 = D_1(K) > 0$ such that

$$|z - x| \geqslant D_1, \quad z \in K, \ x \in \mathbb{R}.$$

Using (6) and the orthonormality of q_n, we get

$$|(\hat{\rho}_\gamma - r_n)(z)| = \left| \frac{1}{(q_n l_n)^2(z)} \int \frac{(q_n l_n)^2(x)}{z - x} \, \mathrm{d}\rho_\gamma \right|$$

$$\leqslant \frac{1}{D_1 |(q_n l_n)^2(z)|}.$$

Now, from Lemma 2.2 and (3) as applied to the sequence $\{l_n\}$, we obtain (7). □

Corollary 1. *Under the assumptions of Theorem* 1

$$r_n \longrightarrow \hat{\rho}_\gamma,$$

uniformly on each compact set of $\mathbb{C}\backslash\mathbb{R}$.

Proof. It is immediate from the fact that the right-hand side of (7) is continuous and negative on $\mathbb{C}\backslash\mathbb{R}$. □

Remark 1. In the case when $\theta = 1$ and $1 < \beta = \gamma$ it is possible to construct examples where there is divergence. For example, taking $m(n) = n$ and $f_\gamma(x) = |x|^\gamma - |x|^{\gamma'}$ with $\gamma' < \gamma$ sufficiently close to γ. For this reason we do not discuss this limiting situation.

Remark 2. When $1 < \gamma < \beta$ and $m(n) = n$ there is always divergence.

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